矩阵求导推导

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1 $\frac{\partial(\boldsymbol{x}^T\boldsymbol{a})}{\partial\boldsymbol{x}}=\frac{\partial(\boldsymbol{a}^T\boldsymbol{x})}{\partial\boldsymbol{x}}=\boldsymbol{a}$

其中，$\boldsymbol{a}$ 为常数向量，$\boldsymbol{a}=(a_1,a_2,\cdots,a_n)^T$。

$$
\begin{aligned}
\frac{\partial(\boldsymbol{x}^T\boldsymbol{a})}{\partial\boldsymbol{x}} &= \frac{\partial(\boldsymbol{a}^T\boldsymbol{x})}{\partial\boldsymbol{x}} \\
&= \frac{\partial(a_1x_1+a_2x_2+\cdots+a_nx_n)}{\partial\boldsymbol{x}} \\
&= \begin{bmatrix} \frac{\partial(a_1x_1+a_2x_2+\cdots+a_nx_n)}{\partial x_1} \\ \frac{\partial(a_1x_1+a_2x_2+\cdots+a_nx_n)}{\partial x_2} \\ \vdots \\ \frac{\partial(a_1x_1+a_2x_2+\cdots+a_nx_n)}{\partial x_n} \end{bmatrix} \\
&= \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \\
&= \boldsymbol{a}
\end{aligned} \tag{1}
$$

2 $\frac{\partial(\boldsymbol{x}^T\boldsymbol{x})}{\partial\boldsymbol{x}} = 2\boldsymbol{x}$

$$
\begin{aligned}
\frac{\partial(\boldsymbol{x}^T\boldsymbol{x})}{\partial\boldsymbol{x}} &= \frac{\partial(x_1^2 + x_2^2 + \cdots + x_n^2)}{\partial\boldsymbol{x}} \\
&= \begin{bmatrix} \frac{\partial(x_1^2+x_2^2+\cdots+x_n^2)}{\partial x_1} \\ \frac{\partial(x_1^2+x_2^2+\cdots+x_n^2)}{\partial x_2} \\ \vdots \\ \frac{\partial(x_1^2+x_2^2+\cdots+x_n^2)}{\partial x_n} \end{bmatrix} \\
&= \begin{bmatrix} 2x_1 \\ 2x_2 \\ \vdots \\ 2x_n \end{bmatrix} \\
&= 2\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\
&= 2\boldsymbol{x}
\end{aligned}
\tag{2}
$$

3 $\frac{\partial(\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x})}{\partial\boldsymbol{x}}=\boldsymbol{A}\boldsymbol{x}+\boldsymbol{A}^T\boldsymbol{x}$

其中，$\boldsymbol{A}_{n\times n}$ 是常数矩阵，$\boldsymbol{A}_{n\times n}=(a_{ij})_{i=1,j=1}^{n,n}$ 。

$$
\begin{aligned}
\frac{\partial\left(\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}\right)}{\partial\boldsymbol{x}}
&=\frac{\begin{array}{c}\partial\left(a_{11}x_1x_1+a_{12}x_1x_2+\cdots+a_{1n}x_1x_n\right.\\ +a_{21}x_2x_1+a_{22}x_2x_2+\cdots+a_{2n}x_2x_n\\ \left.+a_{n1}x_nx_1+a_{n2}x_nx_2+\cdots+a_{nn}x_nx_n\right)\end{array}}{\partial\boldsymbol{x}}\\
&=\begin{bmatrix}(a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n)+(a_{11}x_1+a_{21}x_2+\cdots+a_{n1}x_n)\\ (a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n)+(a_{12}x_1+a_{22}x_2+\cdots+a_{n2}x_n)\\ \vdots\\ (a_{n1}x_1+a_{n2}x_2+\cdots+a_{nn}x_n)+(a_{1n}x_1+a_{2n}x_2+\cdots+a_{nn}x_n)\end{bmatrix}\\
&=\begin{bmatrix}a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n\\ a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n\\ \vdots\\ a_{n1}x_1+a_{n2}x_2+\cdots+a_{nn}x_n\end{bmatrix}+\begin{bmatrix}a_{11}x_1+a_{21}x_2+\cdots+a_{n1}x_n\\ a_{12}x_1+a_{22}x_2+\cdots+a_{n2}x_n\\ \vdots\\ a_{1n}x_1+a_{2n}x_2+\cdots+a_{nn}x_n\end{bmatrix}\\
&=\begin{bmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\ a_{21}&a_{22}&\cdots&a_{2n}\\ \vdots&\vdots&\ddots&\vdots\\ a_{n1}&a_{n2}&\cdots&a_{nn}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix}+\begin{bmatrix}a_{11}&a_{21}&\cdots&a_{n1}\\ a_{12}&a_{22}&\cdots&a_{n2}\\ \vdots&\vdots&\ddots&\vdots\\ a_{1n}&a_{2n}&\cdots&a_{nn}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix}\\
&=\boldsymbol{A}\boldsymbol{x}+\boldsymbol{A}^T\boldsymbol{x}
\end{aligned}
\tag{3}
$$